Strong measure zero subsets of the real line and an infinite game.

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Let $\kappa, \delta, \lambda$ and $\mu$ be cardinal numbers. In a summary of unpublished results by himself and several collaborators, F. Galvin describes the following infinitely long cut-and-choose game: Players ONE and TWO play an inning per positive integer. In the $n$–th inning ONE first chooses a partition of $\kappa$ consisting of fewer than $\delta$ subsets of $\kappa$; let $O_n$ denote the partition chosen by ONE. Then TWO chooses fewer than $\lambda$ of the sets in $O_n$; let $T_n$ denote the sets chosen by TWO. Thus, for each $n$ we have $T_n \subseteq O_n$, $|O_n| < \delta$ and $|T_n| < \lambda$. A play $O_1, T_1, \ldots, O_n, T_n, \ldots$ is won by ONE if

$$|\kappa \setminus \bigcup_{n=1}^{\infty} T_n| \geq \mu.$$ 

The symbol $G(\kappa, \delta, \lambda, \mu)$ denotes this game.

Throughout this paper we assume that $\kappa$ is infinite and $\mu$ is non-zero. Galvin noted that the game $G(\kappa, \delta, \lambda, \mu)$ is equivalent to the game $G(\kappa, \delta, \lambda, 1)$, in the sense that a player has a winning strategy in one of these games if, and only if, the same player has a winning strategy in the other game. Thus, we shall from now on assume that $\mu = 1$, and let $G(\kappa, \delta, \lambda)$ denote the corresponding game.

For functions $f$ and $g$ from $\mathbb{N}$ to $\mathbb{N}$ the symbol $f \prec g$ denotes that $\lim_{n \to \infty} (g(n) - f(n)) = \infty$. This defines a partial order, and the symbol $d$ denotes the cofinality of this partial ordering. The symbol $M$ denotes the collection of first category subsets of the real line, and $\text{cov}(M)$ denotes the minimal cardinality of a family of first category sets which covers the real line. According to the Baire category theorem $\text{cov}(M)$ is uncountable. It is also well–known that $\text{cov}(M) \leq d$. Galvin also showed that these two cardinal numbers can be characterized as follows in terms of the games above:

1. $\kappa < d$ if, and only if, ONE has no winning strategy in the game $G(\kappa, \aleph_1, \aleph_0)$.

2. $\kappa < \text{cov}(M)$ if, and only if, ONE has no winning strategy in the game $G(\kappa, \aleph_1, 2)$, if, and only if, for each $n > 1$ ONE has no winning strategy in $G(\kappa, \aleph_1, n)$. 

1
It appears that the game $G(\kappa, \aleph_0, 2)$ was overlooked. One of the purposes of this note is to show that also this game characterizes one of the important cardinal numbers related to subsets of the real line. In [1] Borel introduced the notion of a strong measure zero set. A subset $X$ of the real line has strong measure zero if there is for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers a sequence $(I_n : n \in \mathbb{N})$ such that each $I_n$ is an open interval, and for each $n$ the length of $I_n$ is less than $\epsilon_n$. We let $\text{SMZ}$ denote the collection of strong measure zero subsets of the real line, and $\text{unif}(\text{SMZ})$ denotes the minimal cardinality for a set of real numbers which does not have strong measure zero. It is easy to see that $\text{unif}(\text{SMZ})$ is uncountable. (Borel conjectured that no uncountable set of real numbers has strong measure zero.) In his study of strong measure zero sets, Rothberger introduced in [2] the property $\text{C}'$: A set $X$ of real numbers has this property if there is for each sequence $(U_n : n \in \mathbb{N})$ of finite open covers of $X$ a sequence $(U_n : n \in \mathbb{N})$ such that for each $n U_n \in U_n$, and $\{U_n : n \in \mathbb{N}\}$ is a cover of $X$.

The relationship between Borel’s strong measure zero sets and Rothberger’s property $\text{C}'$ is as follows (Lemma 3 of [3]):

**Theorem 1 (Rothberger)** A set of real numbers has property $\text{C}'$ if, and only if, each real-valued continuous image of it has strong measure zero.

If we let $\text{unif}(\text{C}')$ denote the minimal cardinality for a set of real numbers which does not have property $\text{C}'$, then Theorem 1 implies that

$$\text{unif}(\text{SMZ}) = \text{unif}(\text{C}')$$

In [3] Rothberger also showed that the Continuum Hypothesis implies that there is a strong measure zero set with the whole real line as continuous image. Thus Borel’s property and Rothberger’s property are not exactly the same.

The following characterization of $\text{unif}(\text{SMZ})$ can be gotten from results in [2] and [3]:

**Theorem 2 (Rothberger)** For an infinite cardinal number $\kappa$ the following are equivalent:

1. $\kappa < \text{unif}(\text{SMZ})$;

2. For every increasing sequence $f$ of positive integers and each subset $F$ of $\Pi_{n \in \mathbb{N}} f(n)$ of cardinality at most $\kappa$, there is a sequence $g$ of positive integers such that for each $h$ in $F$ and for infinitely many $n$, $h(n) = g(n)$.

This result of Rothberger’s and modern results imply that $\text{cov}(\mathcal{M}) \leq \text{unif}(\text{SMZ})$; it is also known to be consistent that equality of these two cardinal numbers is not provable from ZFC. For convenience we let $A(\kappa)$ denote the statement in part 2 of Theorem 2.

Also, Let $B(\kappa)$ denote the statement
For every sequence \((P_n : n \in \mathbb{N})\) where each \(P_n\) is a partition of \(\kappa\) into finitely many disjoint sets, there is a sequence \((P_n : n \in \mathbb{N})\) such that for each \(n\) \(P_n \in P_n\), and \(\cup_{n=1}^{\infty} P_n = \kappa\).

**Theorem 3** For an infinite cardinal number \(\kappa\) the following are equivalent:

1. \(\kappa \leq \text{unif(SMZ)}\);
2. \(\text{A}(\kappa)\);
3. \(\text{B}(\kappa)\);
4. ONE does not have a winning strategy in the game \(G(\kappa, \mathbb{N}_0, 2)\).

**Proof:** The equivalence of 1 and 2 can be gleaned from the cited Rothberger papers. For convenience we give a proof here.

1 \(\Rightarrow\) 2. Let \(f\) be an increasing sequence of positive integers and let \(\mathcal{F}\) be a family of \(\kappa\) elements of \(\Pi_{n \in \mathbb{N}} f(n)\). Considered as a subset of the latter topological space, \(\mathcal{F}\) has property \(C'\). Let \((I_n : n \in \mathbb{N})\) be a partition of \(\mathbb{N}\) into disjoint intervals such that for \(m < n\), each element of \(I_m\) precedes each of \(I_n\). For each \(n\) let \(U_n\) be the open cover

\[ \{[g|_{I_n}] : g \in \Pi_{n \in \mathbb{N}} f(n)\} \]

Let \((Y_n : n \in \mathbb{N})\) be a partition of \(\mathbb{N}\) into pairwise disjoint infinite subsets. For each \(n\) apply \(C'\) to the sequence \((U_m : m \in Y_n)\) of finite open covers of \(\mathcal{F}\). For each \(n\) we find for each \(m \in Y_n\) an \(f_m \in \Pi_{k \in \mathbb{N}} f(k)\) such that \(\{[f_m|_{I_n}] : m \in Y_n\}\) is an open cover of \(\mathcal{F}\).

Define \(g\) by

\[ g(j) = f_m(j) \text{ if } j \in I_m. \]

Then \(g\) has the property that for each \(h \in \mathcal{F}\) there are infinitely many \(m\) such that \(g|_{I_m} = h|_{I_m}\).

2 \(\Rightarrow\) 3. For each \(n\) let \((U^n_k : k < k_n)\) be a partition of \(\kappa\) into finitely many \((k_n)\) disjoint sets. For each \(\alpha < \kappa\) define \(f_\alpha\) such that \(f_\alpha(n)\) is the unique \(k < k_n\) with \(\alpha \in U^k_n\). Then \(\{f_\alpha : \alpha < \kappa\}\) is a family of cardinality at most \(\kappa\) of elements of \(\Pi_{n \in \mathbb{N}} (k_1 + \ldots + k_n)\). Applying \(A(\kappa)\) to this family we find a \(g \in \Pi_{n \in \mathbb{N}} (k_1 + \ldots + k_n)\) such that for each \(\alpha\) there are infinitely many \(j\) with \(f_\alpha(j) = g(j)\). For each \(n\) choose \(j_n = \min\{k_n - 1, g(n)\}\). Then the sequence \((U^n_{j_n} : n \in \mathbb{N})\) is a cover of \(\kappa\).

3 \(\Rightarrow\) 4. Let \(F\) be a strategy for ONE in the game \(G(\kappa, \mathbb{N}_0, 2)\). Using \(F\) define the following array of subsets of \(\kappa\):

1. \(U^\emptyset = \kappa\);
2. \((U_n : n < k_0) = F(U^\emptyset)\), ONE’s first move;
3. For each \(m\), \((U_{(n_2, \ldots, n_m, j)} : j < k_{(n_2, \ldots, n_m)})\) is \(F(U_{(n_1)}, \ldots, U_{(n_1, \ldots, n_m)})\).
Then define a sequence \((U_n : n \in \mathbb{N})\) of finite partitions of \(\kappa\) as follows:

A \(U_1 = \{U(n) : n < k_0\}\);

B \(U_{m+1}\) is a refinement of \(\{U_\tau : \text{length}(\tau) = m + 1\}\) such that for each \(U \in U_{m+1}\) and for each \((n_1, \ldots, n_m)\) with \(n_1 < k_0, n_2 < k(n_1), \ldots, n_m < k(n_1, \ldots, n_{m-1})\), there is a \(j < k(n_1, \ldots, n_m)\) with \(U \subseteq U(n_1, \ldots, n_{m-1}, j)\).

Applying \(\mathcal{B}(\kappa)\) to the sequence \((U_n : n \in \mathbb{N})\) we find for each \(n\) a \(V_n \in U_n\) such that \(\{V_n : n \in \mathbb{N}\}\) covers \(\kappa\). Now choose \(n_1 < k_0\) with \(V_1 = U_{n_1}\). In general, with \(n_1, \ldots, n_j\) selected, choose \(n_{j+1} < k(n_1, \ldots, n_j)\) with \(V_{j+1} \subseteq U(n_1, \ldots, n_{j+1})\). Then the sequence

\[U(n_1), \ldots, U(n_1, \ldots, n_k), \ldots\]

constitute a sequence of consecutive moves by TWO against the strategy \(F\), which defeats \(F\).

References

